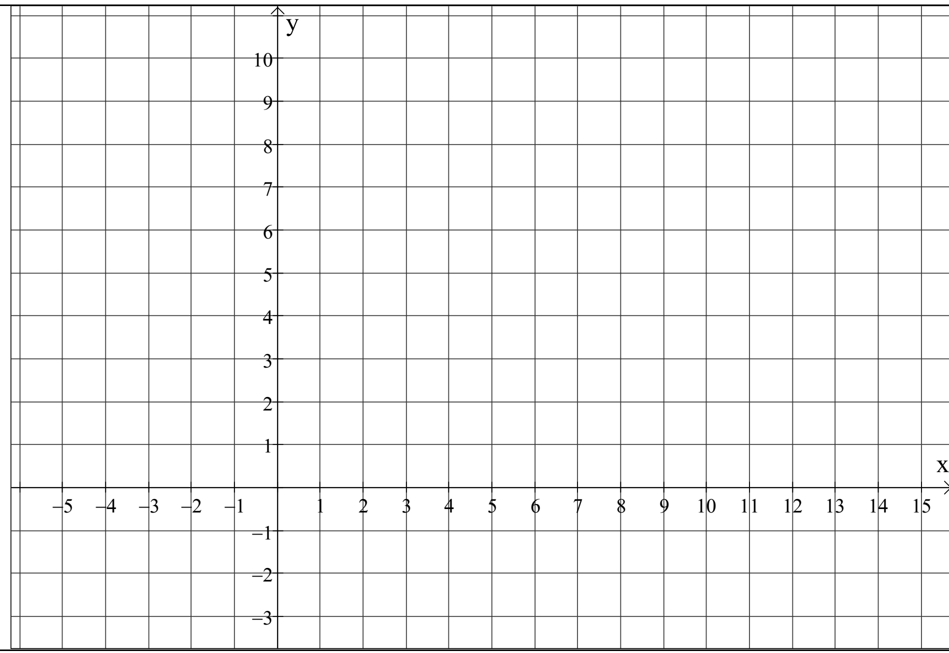
VOLUME REVIEW

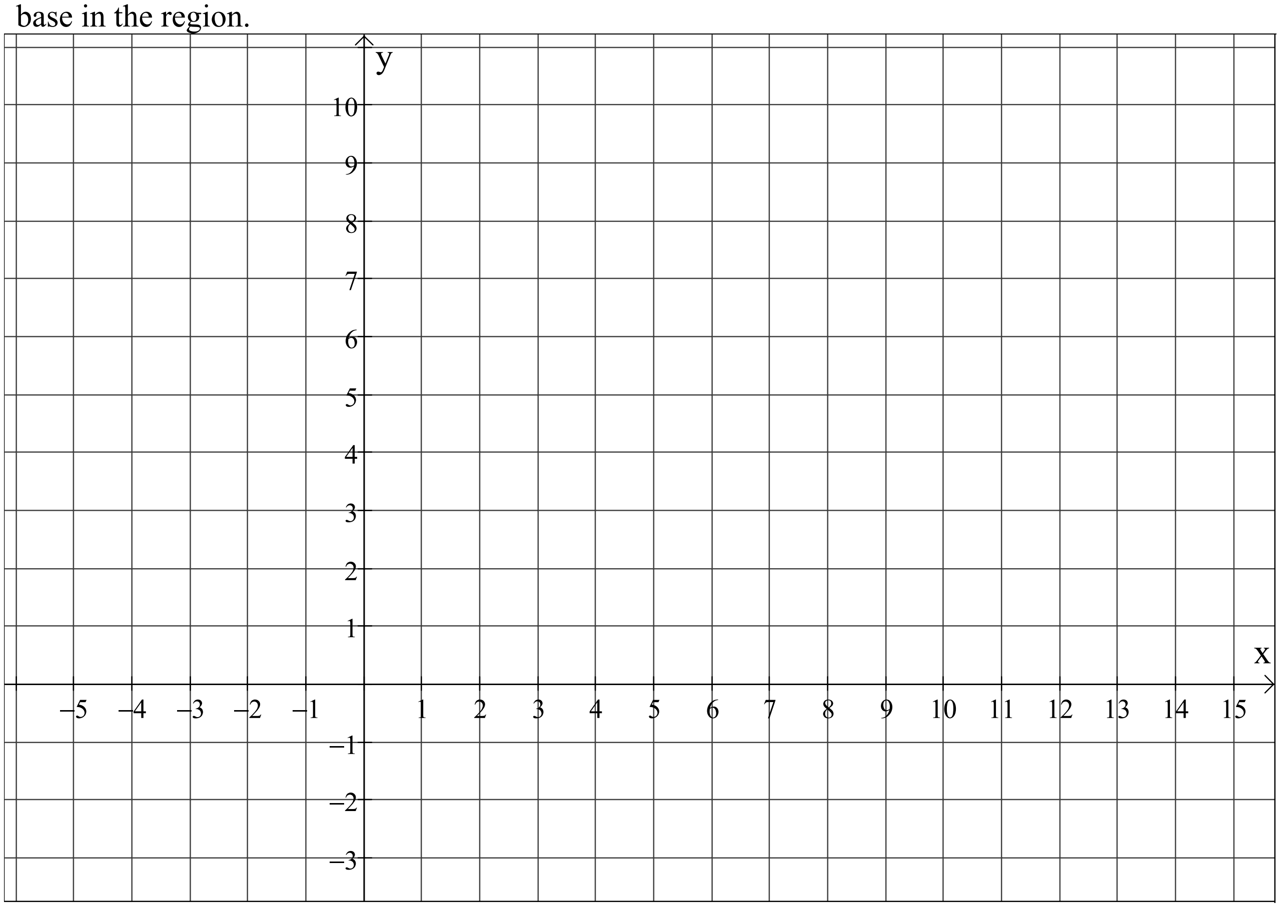
Find the volume of the solid that results when the area of the region enclosed by $y=\sqrt{x}+1$, $x=4$, and $y=1$ is revolved about the ….

1. x–axis
2. y–axis
3. the line $y=1$
4. the line $x=4$
5. the line $y=3$
6. the line $x=-1$
7. the line $y=-1$
8. the line $x=6$
9. the line $y=4$



Find the volume of the solid that results when the area of the region enclosed by $y=\sqrt{x}+1$, $x=4$, and $y=1$

10. has cross sections perpendicular to the x–axis that are squares.
11. has cross sections perpendicular to the x–axis that are semi–circles.
12. has cross sections perpendicular to the x–axis that are rectangles whose height is 5 times the length of its base in the region.



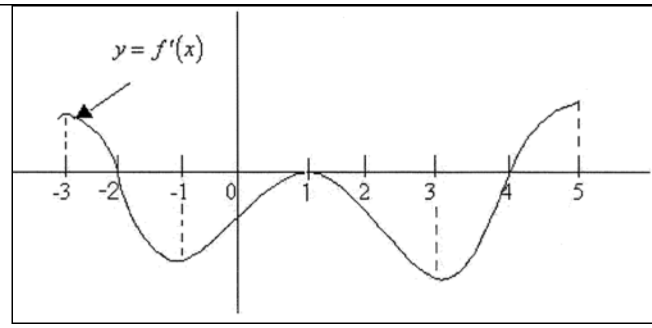
VOLUME REVIEW
ANSWERS

1.	$V=\pi\int_0^4\left[\left(\sqrt{x}+1\right)^2-(1)^2\right]dx=58.643$	(Washer)
2.	$V=\pi\int_1^3\left[(4)^2-\left((y-1)^2\right)^2\right]dy=80.425$	(Washer)
3.	$V=\pi\int_0^4\left[\left(\sqrt{x}+1-1\right)^2\right]dx=25.133$	(Disk)
4.	$V=\pi\int_1^3\left[\left(4-(y-1)^2\right)^2\right]dy=53.617$	(Disk)
5.	$V=\pi\int_0^4\left[(3-1)^2-\left(3-(\sqrt{x}+1)\right)^2\right]dx=41.888$	(Washer)
6.	$V=\pi\int_1^3\left[(4-(-1))^2-\left((y-1)^2-(-1)\right)^2\right]dy=113.935$	(Washer)
7.	$V=\pi\int_0^4\left[\left(\sqrt{x}+1-(-1)\right)^2-\left(1-(-1)\right)^2\right]dx=92.153$	(Washer)
8.	$V=\pi\int_1^3\left[\left(6-(y-1)^2\right)^2-(6-4)^2\right]dy=120.637$	(Washer)
9.	$V=\pi\int_0^4\left[(4-1)^2-\left(4-(\sqrt{x}+1)\right)^2\right]dx=75.398$	(Washer)
10.	$V=\int_0^4(\sqrt{x}+1-1)^2\,dx=8$	{Area = s^2 }
11.	$V=\frac{1}{2}\pi\int_0^4\left(\frac{\sqrt{x}}{2}\right)^2\,dx=3.142$	{Area = $\frac{1}{2}\pi r^2$ }
12.	$V=\int_0^4\left[\left(\sqrt{x}\right)\left(5\sqrt{x}\right)\right]dx=40$	{Area = $5s^2$ }

ANALYZING THE GRAPH OF A DERIVATIVE

PROBLEM #1

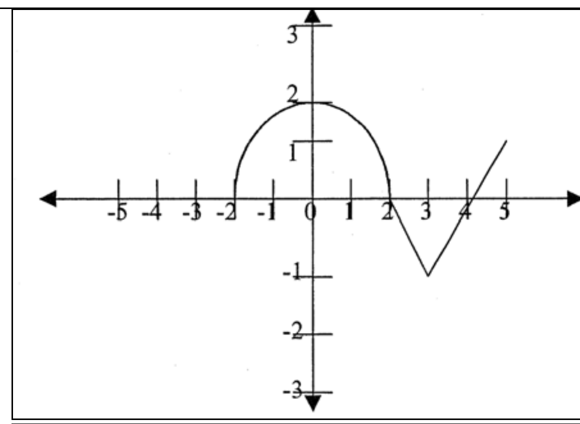
1. For what value(s) of x does *f* have a relative maximum? Why?
2. For what value(s) of x does *f* have a relative minimum? Why?
3. On what intervals is the graph of *f* concave up? Why?
4. On what intervals is *f* increasing? Why?
5. For what value(s) of x does *f* have an inflection point? Why?



PROBLEM #2

The graph of a function f consists of a semicircle and two line segments as shown above. Let g be the function given by $g(x) = \int_0^x f(t)\, dt$

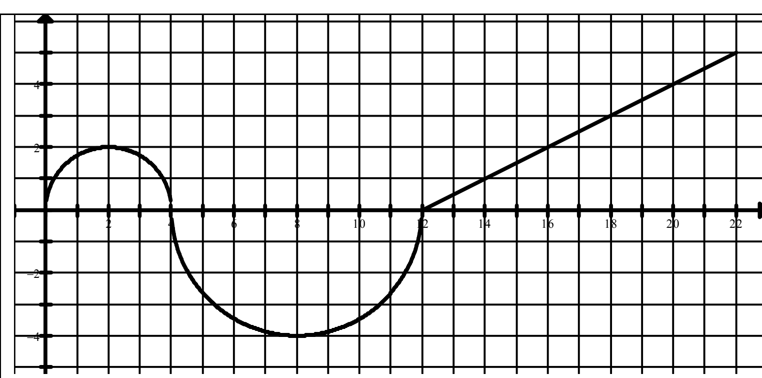
1. Find g(3).
2. For what value(s) of x does *g* have a relative maximum? Why?
3. For what value(s) of x does *g* have a relative minimum? Why?
4. For what value(s) of x does g have an inflection point? Why?
5. Write an equation for the line tangent to the graph of *g* at *x*=3



PROBLEM #3

The graph below shows f ′, the derivative of function f. The graph consists of two semi−circles and one line segment. Horizontal tangents are located at x = 2 and x = 8 and a vertical tangent is located at x = 4.

1. On what intervals is f increasing? Justify your answer.
2. For what values of x does f have a relative minimum? Justify.
3. On what intervals is f concave up? Justify.
4. For what values of x is f ″ undefined?
5. Identify the x−coordinates for all points of inflection of f.
6. For what value of x does f reach its maximum value? Justify.
7. If f (4) = 5, find f(12).



ANALYZING THE GRAPH OF A DERIVATIVE ANSWERS

PROBLEM #1 ANSWERS:

1. at $x=-2$, because $f'(x)$ changes from positive to negative at $x=-2$.
2. at $x=4$, because $f'(x)$ changes from negative to positive at $x=4$.
3. (−1,1) and (3,5) because $f'(x)$ is increasing on these intervals (thus $f''(x)>0$).
4. (−3,−2) and (4,5) because $f'(x)>0$ on these intervals.
5. at x = −1, x = 1 and x = 3, because $f''(x)$ changes signs at these values of x.

PROBLEM #2 ANSWERS:

1. $g(3)=\int_0^3 f(t)\,dt=\left(\frac{1}{4}\right)(\pi)(2^2)+\left(\frac{1}{2}\right)(1)(-1)=\pi-\frac{1}{2}$
2. At $x=2$, because $g'(x)=f(x)$ changes from positive to negative at $x=2$.
3. At $x=4$, because $g'(x)=f(x)$ changes from negative to positive at $x=4$.
4. At $x=0$ and $x=3$, because $g''(x)=f'(x)$ changes from positive to negative at $x=0$ and changes from negative to positive at $x=3$.
5. $g(3)=\pi-\frac{1}{2}$ and $g'(3)=f(3)=-1$ $\therefore$ $y-\left(\pi-\frac{1}{2}\right)=-1(x-3)$

PROBLEM #3 ANSWERS:

1. (0,4) and (12,22) because $f'(x)>0$ for these values of x.
2. At $x=12$ because $f'(x)$ changes from negative to positive at $x=12$.
3. (0,2) (8,12) and (12,22) because $f''(x)>0$ for these values of x. [Or, $f'(x)$ is increasing for these values of x.]
4. At $x=4$ and $x=12$
5. At $x=2$ and $x=8$
6. At $x=4$ because $f(x)$ is increasing on (0,4) [$f'(x)>0$] and $f(x)$ is decreasing on (4,12) [$f'(x)<0$] and $f(x)$ is increasing on (12,22) [$f'(x)>0$]. Thus, max can occur at $x=4$ or at $x=22$ On the interval (4,12), $f(x)$ decreases by: $\int_4^{12} f'(x)\,dx=8\pi$ On the interval (12,22), $f(x)$ increases by: $\int_{12}^{22} f'(x)\,dx=25$ Because $8\pi>25$, $f(x)$ decreases by $8\pi-25$ on the interval (4,22). Thus, maximum occurs at $x=4$.
7. $f(12)-f(4)=\int_4^{12} f'(x)\,dx \Rightarrow f(12)=f(4)+\int_4^{12} f'(x)\,dx = 5-8\pi$

TABLE OF VALUES (CALCULATOR)
PIE PROBLEM

1. Let $y(t)$ represent the temperature of a pie that has been removed from a 450°F oven and left to cool in a room with a temperature of 72°F, where y is a differentiable function of t. The table below shows the temperature recorded every five minutes.

t (min)	0	5	10	15	20	25	30
$y(t)$ $(^\circ\text{F})$	450	388	338	292	257	226	200

A) Use data from the table to find an approximation for $y'(18)$, and explain the meaning of $y'(18)$ in terms of the temperature of the pie. Show the computations that lead to your answer, and indicate units of measure.

B) Use data from the table to find the value of $\int_{10}^{25} y'(t)\,dt$, and explain the meaning of $\int_{10}^{25} y'(t)\,dt$ in terms of the temperature of the pie. Indicate units of measure.

C) A model for the temperature of the pie is given by the function: $W(t) = 72 + 380e^{-0.036t}$ where t is measured in minutes and $W(t)$ is measured in degrees Fahrenheit $(^\circ\text{F})$. Use the model to find the value of $W'(18)$. Indicate units of measure.

D) Use the model given in part (c) to find the time at which the temperature of the pie is 300°F.

TABLE OF VALUES (ANSWERS)
PIE PROBLEM

A) Use data from the table to find an approximation for $y'(18)$, and explain the meaning of $y'(18)$ in terms of the temperature of the pie. Show the computations that lead to your answer, and indicate units of measure.

$$y'(18) \approx \frac{257-292}{20-15} = \boxed{-7^\circ \frac{\text{F}}{\text{min.}}}$$

When $t = 18$ minutes, the temperature of the pie is decreasing at a rate of approximately 7°F per minute.

B) Use data from the table to find the value of $\int_{10}^{25} y'(t)\,dt$, and explain the meaning of $\int_{10}^{25} y'(t)\,dt$ in terms of the temperature of the pie. Indicate units of measure.

$$\int_{10}^{25} y'(t)\,dt = y(25) - y(10) = \boxed{-112^\circ\text{F}}$$

From $t = 10$ minutes to $t = 25$ minutes, the temperature of the pie dropped 112°F.

C) A model for the temperature of the pie is given by the function: $W(t) = 72 + 380e^{-0.036t}$ where t is measure in minutes and $W(t)$ is measured in degrees Fahrenheit $(^\circ\text{F})$. Use the model to find the value of $W'(18)$. Indicate units of measure.

$W'(18) = \boxed{-7.156^\circ\text{F}}$ per minute.

D) Use the model given in part (c) to find the time at which the temperature of the pie is 300°F.

$W(t) = 300$ when $t = \boxed{14.190 \text{ minutes}}$.

TABLE OF VALUES
SUGAR MILL

2. Let $y(t)$ represent the population of the town of Sugar Mill over a 10-year period, where y is a differentiable function of t. The table below shows the population recorded every two years.

t (yrs)	0	2	4	6	8	10
y(people)	2500	2912	3360	3815	4330	4875

A) Use data from the table to find an approximation for $y'(7)$, and explain the meaning of $y'(7)$ in terms of the population of Sugar Mill. Show the computations that lead to your answer.

B) Use data from the table to approximate the average population of Sugar Mill over the time interval $0 \le t \le 10$ by using a left Reimann sum with five equal subintervals. Show the computations that lead to your answer.

C) A model for the population of another town, Pine Grove, over the same 10-year period is given by the function $P(t) = (2t+50)^2$, where t is measured in years and P(t) is measured in people. Use the model to find the value of P'(7).

D) Use the model given in part (c) to find the value of $\frac{1}{10}\int_0^{10} P(t)\,dt$. Explain the meaning of this integral expression in terms of the population of Pine Grove.

TABLE OF VALUES (ANSWERS)
SUGAR MILL

2. Let y(t) represent the population of the town of Sugar Mill over a 10-year period, where y is a differentiable function of t. The table below shows the population recorded every two years.

t (yrs)	0	2	4	6	8	10
$y(\text{people})$	2500	2912	3360	3815	4330	4875

A) Use data from the table to find an approximation for y'(7), and explain the meaning of y'(7) in terms of the population of Sugar Mill. Show the computations that lead to your answer.

$$y'(7) = \frac{4330 - 3815}{8 - 6} = \boxed{257.5}$$

When $t = 7$ years, the population of Sugar Mill is increasing at a rate of approximately 257.5 people per year.

B) Use data from the table to approximate the average population of Sugar Mill over the time interval $0 \le t \le 10$ By using a left Reimann sum with five equal subintervals. Show the computations that lead to your answer.

$$\text{Average Population} = \frac{1}{10}\int_0^{10} y(t)\,dt = \frac{1}{10}\big(2(2500 + 2912 + 3360 + 3815 + 4330)\big) = \boxed{3383.4}$$

so the average population over the 10 – year period was approximately 3383.4 people.

C) A model for the population of another town, Pine Grove, over the same 10-year period is given by the function $P(t) = (2t + 50)^2$, where t is measured in years and P(t) is measured in people. Use the model to find the value of P'(7).

$$\boxed{P'(7) = 256 \text{ people per year}}$$

D) Use the model given in part (c) to find the value of $\frac{1}{10}\int_0^{10} P(t)dt$. Explain the meaning of this integral expression in terms of the population of Pine Grove.

$\frac{1}{10}\int_0^{10} P(t)dt = \boxed{3633.333 \text{ people}}$. This means that the average population of Pine Grove over the 10 – year period was approximately 3633.333 people.

TABLE OF VALUES
BOWL OF SOUP

3. A bowl of soup is place on the kitchen counter to cool. Let $T(x)$ represent the temperature of the soup at time x, where T is a differentiable function of x. The temperature of the soup at selected times is given in the table below.

x (min)	0	4	7	12
$T(x)\left({}^{\circ}F\right)$	108	101	99	95

A) Use data from the table to find:

$$\int_0^{12} T'(x)\,dx$$

Explain the meaning of this definite integral in terms of the temperature of the soup.

B) Use data from the table to find the average rate of change of $T(x)$ over the time interval $x = 4$ to $x = 7$

C) Explain the meaning of:

$$\frac{1}{12}\int_0^{12} T(x)\,dx$$

In terms of the temperature of the soup, and approximate the value of this integral expression by using a trapezoidal sum with three subintervals

TABLE OF VALUES (ANSWERS)
BOWL OF SOUP

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x (min)	0	4	7	12
$T(x)\left({}^\circ F\right)$	108	101	99	95

A) Use data from the table to find:

$$\int_0^{12} T'(x)\,dx$$

Explain the meaning of this definite integral in terms of the temperature of the soup.

$$\int_0^{12} T'(x)\,dx = T(12) - T(6) = \boxed{-13^\circ F}$$

From $x = 0$ to $x = 12$ minutes, the temperature of the soup dropped $13^\circ F$

B) Use data from the table to find the average rate of change of $T(x)$ over the time interval $x = 4$ to $x = 7$

$$\text{Average rate of change} = \frac{T(7) - T(4)}{7 - 4} = \frac{99 - 101}{3} = -\frac{2^\circ}{3}\ F/\min$$

C) Explain the meaning of:

$$\frac{1}{12}\int_0^{12} T(x)\,dx$$

In terms of the temperature of the soup, and approximate the value of this integral expression by using a trapezoidal sum with three subintervals

$\frac{1}{12}\int_0^{12} T(x)\,dx$ represents the average temperature of the soup over the 12-minute period and is approximately equal to:

$$\frac{1}{12}\left(\frac{1}{2}(4)(108+101) + \frac{1}{2}(3)(101+99) + \frac{1}{2}(5)(99+95)\right) = \boxed{100.25^\circ F}$$

TABLE OF VALUES
WATER INTO A TANK

4. The rate at which water is being pumped into a tank is given by the continuous, increasing function $R(t)$. A table of values of $R(t)$, for the time interval $0 \le t \le 20$ minutes, is shown below

t (min)	0	4	9	17	20
$R(t)$ (gal/min)	25	28	33	42	46

A) Use a right Riemann sum with four subintervals to approximate the value of:

$$\int_0^{20} R(t)\,dt$$

Is your approximation greater or less than the true value? Give a reason for your answer.

B) A model for the rate at which water is being pumped into the tank is given by the function:

$$W(t) = 25e^{0.03t}$$

where t is measured in minutes and $W(t)$ is measured in gallons per minute. Use the model to find the average rate at which water is being pumped into the tank from $t = 0$ to $t = 20$ minutes.

C) The tank contained 100 gallons of water at time $t = 0$. Use the model given in part (b) to find the amount of water in the tank at t = 20 minutes

TABLE OF VALUES (ANSWERS)
WATER INTO A TANK

4. The rate at which water is being pumped into a tank is given by the continuous, increasing function $R(t)$. A table of values of $R(t)$, for the interval $0 \le t \le 20$ minutes, is shown below

t (min)	0	4	9	17	20
$R(t)$ (gal/min)	25	28	33	42	46

A) Use a right Riemann sum with four subintervals to approximate the value of:

$$\int_0^{20} R(t)\,dt$$

Is your approximation greater or less than the true value? Give a reason for your answer.

$$\int_0^{20} R(t)\,dt \approx (4)(28)+(5)(33)+(8)(42)+(3)(46) = \boxed{751 \text{ gallons}}$$

Since R is positive, this is an estimate of the amount of water pumped into the tank during the 20-minute period. Since R increases on $0 < t < 20$, the right Riemann sum approximation of 751 gallons is greater than $\int_0^{20} R(t)\ dt$.

B) A model for the rate at which water is being pumped into the tank is given by the function:

$$W(t) = 25e^{0.03t}$$

Where t is measured in minutes and $W(t)$ is measured in gallons per minute. Use the model to find the average rate at which water is being pumped into the tank from $t = 0$ to $t = 20$ minutes.

$$\text{Average Rate} = \frac{1}{20}\int_0^{20} W(t)\,dt = \boxed{34.255 \text{ gal/min}}$$

C) The tank contained 100 gallons of water at time $t = 0$. Use the model given in part (b) to find the amount of water in the tank at t = 20 minutes

$$100 + \int_0^{20} W(t)\,dt = \boxed{785.099 \text{ gallons}}$$

TABLE OF VALUES
CAR VELOCITY

5. Car A has positive velocity $v_A(t)$ as it travels on a straight road, where v_A is a differentiable function of t. The velocity is recorded for selected values over the time interval $0 \le t \le 10$ seconds, as shown in the table below.

t (sec)	0	2	5	7	10
$v_A(t)$ (ft/sec)	1	9	36	61	115

A) Use data from the table to approximate the acceleration of Car A at t = 8 seconds. Indicate units of measure.

B) Use data from the table to approximate the distance traveled by Car A over the interval $0 \le t \le 10$ seconds by using a trapezoidal sum with four subintervals. Show the computations that lead to your answer, and indicate units of measure.

C) Car B travels along the same road with an acceleration of $a_B(t) = 2t + 2 \quad ft/\sec^2$. At time t = 3 seconds, the velocity of car B is 11 ft/sec. Which car is traveling faster at time t = 7 seconds? Explain your answer.

TABLE OF VALUES (ANSWERS)
CAR VELOCITY

5. Car A has positive velocity $v_A(t)$ as it travels on a straight road, where v_A is a differentiable function of t. The velocity is recorded for selected values over the time interval $0 \le t \le 10$ seconds, as shown in the table below.

t (sec)	0	2	5	7	10
$v_A(t)$ (ft/sec)	0	9	36	61	115

A) Use data from the table to approximate the acceleration of Car A at t = 8 seconds. Indicate units of measure.

Let $a_A(t)$ be the acceleration of Car A at time t. Then:

$$a_A(8) \approx \frac{v_A(10) - v_A(7)}{10-7} = \frac{115-61}{3} = 18\frac{ft/\sec}{\sec} = \boxed{18\frac{ft}{\sec^2}}$$

B) Use data from the table to approximate the distance traveled by Car A over the interval $0 \le t \le 10$ seconds by using a trapezoidal sum with four subintervals. Show the computations that lead to your answer, and indicate units of measure.

Distance: $$\int_0^{10} v_A dt \approx \frac{1}{2}\left[(2)(1+9)+(3)(9+36)+(2)(36+61)+(3)(61+115)\right] = \boxed{437.5\ ft}$$

C) Car B travels along the same road with an acceleration of $a_B(t) = 2t + 2\ ft/\sec^2$. At time t = 3 seconds, the velocity of car B is 11 ft/sec. Which car is traveling faster at time t = 7 seconds? Explain your answer.

Let be the velocity of Car B at time t. Then: $v_B(t) = \int (2t+2)\,dt = t^2 + 2t + C$

At $t = 3$, we have $11 = 9 + 6 + C$, so that $C = -4$ and $\boxed{v_B(t) = t^2 + 2t - 4}$.

Hence, $v_B(7) = 59 < 61 = v_A(7)$. **We conclude that Car A is traveling faster at time $t = 7$ seconds.**

TABLE OF VALUES (CALCULATOR)
WATER TEMPERATURE

t (days)	$W(t)$ (°C)
0	20
3	25
6	28
9	27
12	22
15	19

6. The temperature, in degrees Celsius (°C), of the water in a pond is a differentiable function W of time t. The table above shows the water temperature as recorded every 3 days over a 15-day period.

1) Use data from the table to find the average change in the water temperature for the 15-day period.

2) Use data from the table to find an approximation for $W'(12)$. Show the computations that lead to your answer. Indicate units of measure.

3) Approximate the average temperature, in degrees Celsius, of the water over the time interval $0 \le t \le 15$ days by using a trapezoidal approximation with subintervals of length $\Delta t = 3$ days.

4) A student proposes the function P, given by $P(t) = 20 + 10te^{(-t/3)}$, as a model for the temperature of the water in the pond at time t, where t is measured in days and $P(t)$ is measured in degrees Celsius. Find $P'(12)$. Using appropriate units, explain the meaning of your answer in terms of water temperature.

5) Use the function P defined in part (4) to find the average value, in degrees Celsius, of $P(t)$ over the time interval $0 \le t \le 15$ days.

6) Will $W'(t) = 0$ during the 15-day period? Why or why not?

TABLE OF VALUES (ANSWERS)
WATER TEMPERATURE

1.	$\frac{W(15)-W(0)}{15-0}=\frac{19-20}{15}=\boxed{\frac{-1}{15}\ \frac{°\text{C}}{\text{day}}}$
2.	$\frac{W(15)-W(9)}{15-9}=\frac{19-27}{6}=\boxed{\frac{-4}{3}\ \frac{°\text{C}}{\text{day}}}$ or: $\frac{W(15)-W(12)}{15-12}=\frac{19-22}{3}=\boxed{-1\ \frac{°\text{C}}{\text{day}}}$ or: $\frac{W(12)-W(9)}{12-19}=\frac{22-27}{3}=\boxed{\frac{-5}{3}\ \frac{°\text{C}}{\text{day}}}$
3.	$\frac{1}{15}\bullet\frac{3}{2}\left[20+2(25)+2(28)+2(27)+2(22)+19\right]=\boxed{24.3\ °\text{C}}$
4.	$P'(t)=10e^{(-t/3)}+10te^{(-t/3)}\left(\frac{-1}{3}\right)\ \Rightarrow\ P'(12)=10e^{(-4)}-40e^{(-4)}\ =\boxed{\frac{-30}{e^4}\ \frac{°\text{C}}{\text{day}}}$ Remember, Calc Derivative: $P'(12)\approx\boxed{-0.549}$ or Math 8, y_1, x, $12\approx\boxed{-0.549}$ This means that the temperature is decreasing at the rate of $30e^{-4}\ \frac{°\text{C}}{\text{day}}$ at $t=12$ days
5.	$\frac{1}{15}\int_0^{15}20+10te^{(-t/3)}dt\approx\boxed{25.757\ °\text{C}}$
6.	Yes. By INTERMEDIATE VALUE THEOREM, $W(0)=20$ and somewhere $12<\text{t}<15$, $W(t)=20$ Thus, by MEAN VALUE THEOREM (OR ROLLES) $W'(t)=0$ somewhere $0<\text{t}<15$

TABLE OF VALUES (CALCULATOR)
THE WIRE

Distance x (cm)	0	1	5	6	8
Temperature $T(x)$ $(^\circ\text{C})$	100	89	73	64	51

7. A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius $(^\circ\text{C})$, of the wire x cm from the heated end. The function T is decreasing and twice differentiable.

1. Estimate $T'(7)$. Show the work that leads to your answer. Indicate units of measure.

2. Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with the four subintervals indicated by the data in the table. Indicate units of measure.

3. Find $\int_0^8 T'(x)\,dx$ and indicate units of measure. Explain the meaning of $\int_0^8 T'(x)\,dx$ in terms of the temperature of the wire.

4. Are the data in the table consistent with the assertion that $T''(x) > 0$ for every x in the interval $0 < x < 8$? Explain your answer.

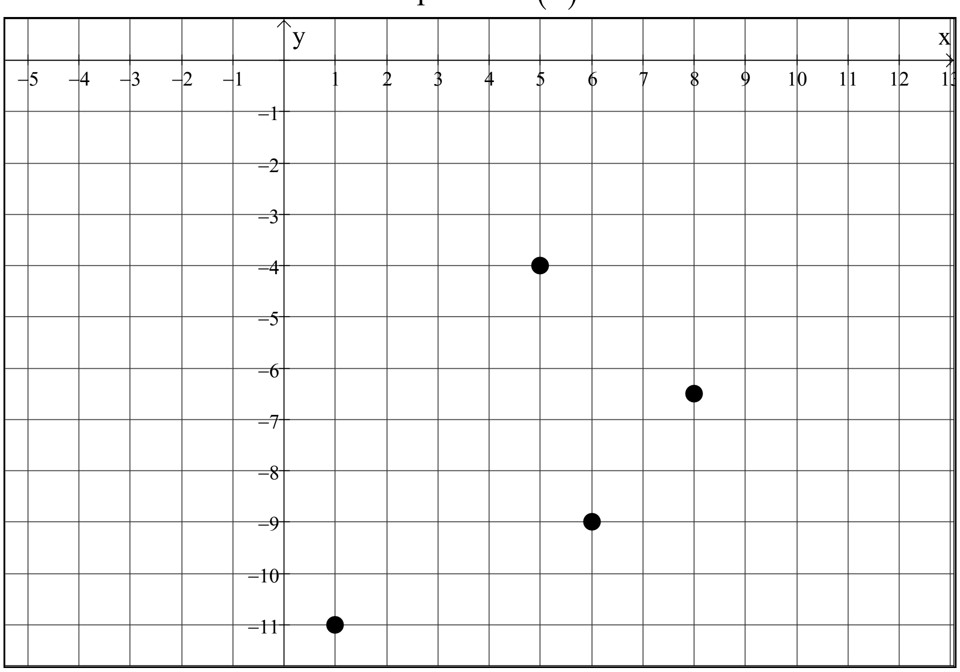
TABLE OF VALUES (ANSWERS)
THE WIRE

1.	$\dfrac{T(8)-T(6)}{8-6}=\dfrac{51-64}{2}=\dfrac{-13}{2}\ ^{\circ}\text{C/cm}$
2.	$\dfrac{1}{8}\displaystyle\int_0^8 T(x)\,dx=\dfrac{1}{8}\left[\dfrac{100+89}{2}(1)+\dfrac{89+73}{2}(4)+\dfrac{73+64}{2}(1)+\dfrac{64+51}{2}(2)\right]\approx 75.25\ ^{\circ}\text{C}$
3.	$\displaystyle\int_0^8 T'(x)\,dx=T(8)-T(0)=51-100=-49\ ^{\circ}\text{C}$ The temperature drops $49\ ^{\circ}\text{C}$ from the heated and of the wire to the other end of the wire.
4.	No. The MVT guarantees that $T''(x)<0$ for some value of x on the interval $1<x<6$ (See graph of $T'(x)$ below)

4.

x	$T(x)$	$T'(x)$ (estimate)
1	89	$\dfrac{89-100}{1-0}=-11$
5	73	$\dfrac{73-89}{5-1}=-4$
6	64	$\dfrac{64-73}{6-5}=-9$
8	51	$\dfrac{51-64}{8-6}=-6.5$

Graph of $T'(x)$



RATES OF CHANGE

PROBLEM #1 (CALCULATOR)

Traffic flow is defined as the rate at which cars pass through an intersection, measured in cars per minute. The traffic flow at a particular intersection is modeled by the function *F* defined by:

$$F(t) = 82 + 4\sin\left(\frac{t}{2}\right) \quad \text{for } 0 \le t \le 10$$

Where $F(t)$ is measured in cars per minute and t is measured in minutes.

1. To the nearest whole number, how many cars pass through the intersection over the 10−minute period?

2. Is the traffic flow increasing or decreasing at $t = 5$? Justify.

3. What is the average value of the traffic flow over the time interval $3 \le t \le 7$? Indicate units of measure.

4. What is the average rate of change of the traffic flow over the time interval $3 \le t \le 7$? Indicate units of measure.

5. At what time, t, is the traffic flow the greatest? What is the greatest flow?

PROBLEM #2 (CALCULATOR)

A water tank at Camp Diamond Bar holds 1200 gallons of water at time $t = 0$. During the time interval $0 \le t \le 12$ hours, water is pumped into the tank at the rate:

$$W(t) = 95\sqrt{t}\sin^2\left(\frac{t}{6}\right) \text{ gallons per hour.}$$

During the same time interval, water is removed from the tank at the rate

$$R(t) = 275\sin^2\left(\frac{t}{3}\right) \text{ gallons per hour.}$$

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1. Is the amount of water in the tank increasing at time $t = 5$? Why or why not?

2. To the nearest whole number, how many gallons of water are in the tank at time $t = 12$?

3. At what time, t, for $0 \le t \le 12$, is the amount of water in the tank at an absolute minimum? Show the work that leads to your conclusion.

4. For $t > 12$, no water is pumped into the tank, but water continues to be removed at the rate $R(t)$ until the tank become empty. Let k be the time at which the tank becomes empty. Write, but do not solve, an equation involving an integral expression that can be used to find the value of k.

5. What is the average rate of change in the amount of water in tank for $0 \le t \le 12$ hours?

RATES OF CHANGE ANSWERS

PROBLEM #1 ANSWERS:

1. 826 cars
2. Decreasing. Because $F'(5) < 0$ $\quad \left[F'(5) \approx -1.602\right]$
3. $\frac{1}{7-3}\int_3^7 F(t)\,dt \;=\; 84.014$ cars/minute
4. $\frac{F(7)-F(3)}{7-3} = -1.348$ cars/minute^2
5. At $t = \pi$ minutes, 86 cars per minute.

PROBLEM #2 ANSWERS:

1. No. the amount of water is not increasing at $t = 5$ because $W(5) - R(5) = -156.0998 < 0$
2. 1643 gallons $\quad \left[1200 + \int_0^{12} W(t) - R(t)\,dt \;\approx\; 1642.630\right]$
3. At $t = 6.495$ Because: at $t = 0$, there are 1200 gallons of water in the tank at $t = 6.495$, there are 525.242 gallons of water in the tank. at $t = 12$, there are 1642.630 gallons of water in the tank.
4. $\int_{12}^{k} R(t)\,dt \;=\; 1643$
5. $\frac{1643-1200}{12-0} = 36.917$ gallons/hour

PARTICLE PROBLEMS

PROBLEM #1 (NO CALCULATOR)

A particle moves along the x−axis with the velocity at time $t \geq 0$ given by $v(t) = -1 + e^{1-t}$.

1. Find the acceleration of the particle at time $t = 3$.

2. Is the speed of the particle increasing at time $t = 3$? Give a reason for your answer.

3. Find all values of t at which the particle changes direction. Justify your answer.

4. What is the average velocity of the particle over the interval $0 \leq t \leq 3$?

5. Find the total distance traveled by the particle over the interval $0 \leq t \leq 3$?

PROBLEM #2 (CALCULATOR)

A particle moves along the y−axis so that its velocity v at time $t \geq 0$ is given by $v(t) = 1 - \tan^{-1}\left(e^{t}\right)$ and $y(0) = -1$

1. Find the acceleration of the particle at time $t = 2$.

2. Is the speed of the particle increasing or decreasing at time $t = 2$? Give a reason for your answer.

3. Find the time $t \geq 0$ at which the particle reaches its highest point. Justify your answer.

4. Find the position of the particle at time $t = 2$. Is the particle moving toward the origin or away from the origin at time $t = 2$? Justify your answer.

5. Find the total distance traveled by the particle over the interval $0 \leq t \leq 3$?

PARTICLE PROBLEMS
ANSWERS

PROBLEM #1 ANSWERS:

1. $a(t)=v'(t)=-e^{1-t} \quad \Rightarrow \quad a(3)=-e^{1-3}=-e^{-2}$
2. $v(3)=-1+e^{-2}<0$ and $a(3)<0 \quad \therefore$ Speed is increasing
3. Particle changes direction at $t=1$ because $v(t)=0$ when $1=e^{1-t}$, so $t=1$. $v(t)$ changes from positive to negative at $t=1$
4. AV = $\dfrac{1}{3-0}\int_0^3 -1+e^{1-t}\,dt \;=\; \dfrac{1}{3}\left[-t-e^{1-t}\right]_0^3 \;=\; \dfrac{1}{3}\left[\left(-3-e^{-2}\right)-\left(0-e\right)\right] \;=\; \dfrac{1}{3}\left(-3-e^{-2}+e\right)$
5. TD $=\int_0^1 -1+e^{1-t}dt-\int_1^3 -1+e^{1-t}dt=\left[-t-e^{1-t}\right]_0^1-\left[-t-e^{1-t}\right]_1^3=$ $\left[\left(-1-e^0\right)-\left(0-e\right)\right]-\left[\left(-3-e^{-2}\right)-\left(-1-e^0\right)\right]=\left[-2+e-\left(-1-e^{-2}\right)\right]=-1+e+e^{-2}$

PROBLEM #2 ANSWERS:

1. $a(2)=v'(2)=-0.133$
2. $v(2)=-0.436$ Speed is increasing since $a(2)$ and $v(2)$ are both negative.
3. $v(t)=0$ when $t\approx 0.443$. $y(t)$ must have a maximum at $t\approx 0.443$ because $y'(t)$ changes from positive to negative at $t\approx 0.443$. Maximum doesn't occur at endpoints because $y(t)$ increases on the interval (0,0.443) and $y(t)$ decreases when $t\geq 0.443$.
4. $y(2)=-1+\int_0^2 v(t)\,dt=-1.361$. The particle is moving away from the origin since $v(2)<0$ and $y(2)<0$.
5. TD $=\int_0^3 \lvert v(t)\rvert\,dt=0.940$

IMPLICIT DIFFERENTIATION

PROBLEM #1 (NO CALCULATOR)

Consider the curve $x^2 y - x^3 y = 1$

1. Use implicit differentiation to show that $\dfrac{dy}{dx} = \dfrac{y(3x-2)}{x(1-x)}$

2. Find the equation of all horizontal tangent lines.

3. Find the equation of all vertical tangent lines.

4. Find the equation of the tangent line(s) at $x = 2$.

5. Using the tangent line at $x = 2$, approximate $y(2.1)$.

6. Is the curve increasing or decreasing at $x = \dfrac{1}{2}$? Justify your answer.

7. Is the curve concave up or down at $x = \dfrac{1}{2}$? Justify your answer.

8. Would a tangent line approximation <u>overestimate</u> or <u>underestimate</u> at $x = \dfrac{1}{2}$? Why?

PROBLEM #2 (NO CALCULATOR)

Consider the curve $xy^2 - x^2 y = 2$

1. Use implicit differentiation to show that $\dfrac{dy}{dx} = \dfrac{y(2x-y)}{x(2y-x)}$

2. Find the equation of all horizontal tangent lines.

3. Find the equation of all vertical tangent lines.

4. Is the curve increasing or decreasing at $(1,-1)$? Justify your answer.

5. Is the curve concave up or down at $(1,-1)$? Justify your answer.

PROBLEM #1 ANSWERS:

2. $y=\frac{27}{4}$ {This occurs when $x=\frac{2}{3}$}	5. $\frac{-1}{5}$
3. $x=0$ or $x=1$	6. Decreasing because $\frac{dy}{dx}=-16$ at $x=\frac{1}{2}$
4. $y+\frac{1}{4}=\frac{1}{2}(x-2)$	7. Concave up because $\frac{d^2y}{dx^2}=128$ at $x=\frac{1}{2}$
8. Underestimate because curve is concave up at $x=\frac{1}{2}$	

PROBLEM #2 ANSWERS:

1.

$$y^2+x(2y)\left(\frac{dy}{dx}\right)-[2xy+]=0 \Rightarrow y^2+(2xy)\left(\frac{dy}{dx}\right)-2xy-x^2\left(\frac{dy}{dx}\right)=0$$

$$\frac{dy}{dx}\left[2xy-x^2\right]=2xy-y^2 \Rightarrow \frac{dy}{dx}=\frac{2xy-y^2}{2xy-x^2}=\frac{y(2x-y)}{x(2y-x)}$$

2.

$$\frac{dy}{dx}=\frac{y(2x-y)}{x(2y-x)}=\frac{0}{1} \Rightarrow y(2x-y)=0 \Rightarrow y=0 \text{ or } 2x-y=0 \Rightarrow y=0 \text{ or } y=2x$$

If $y=0$, then $xy^2-x^2y=2 \Rightarrow x(0)^2-x^2(0)=2 \Rightarrow 0=2$ BAD !!!!!

If $y=2x$, then $x(2x)^2-x^2(2x)=2 \Rightarrow 4x^3-2x^3=2 \Rightarrow 2x^3=2 \Rightarrow x^3=1 \Rightarrow x=1$

If $x=1$, then $xy^2-x^2y=2 \Rightarrow y^2-y=2 \Rightarrow y^2-y-2=0 \Rightarrow (y-2)(y+1)=0 \Rightarrow y=2$ or $y=-1$

3.

$$\boxed{\frac{dy}{dx}=\frac{y(2x-y)}{x(2y-x)}=\frac{1}{0} \Rightarrow x(2y-x)=0 \Rightarrow x=0 \text{ or } 2y-x=0 \Rightarrow x=0 \text{ or } x=2y}$$

If $x=0$, then $xy^2-x^2y=2 \Rightarrow 0y^2-0y=2 \Rightarrow 0=2$ BAD !!!!!

If $x=2y$, then $(2y)y^2-(2y)^2y=2 \Rightarrow 2y^3-4y^3=2 \Rightarrow -2y^3=2 \Rightarrow y^3=-1 \Rightarrow y=-1$

If $y=-1$, then $xy^2-x^2y=2 \Rightarrow x(-1)^2-x^2(-1)=2 \Rightarrow x^2+x-2=0 \Rightarrow (x+2)(x-1)=0 \Rightarrow x=-2$ or $x=1$

4. Increasing because $\frac{dy}{dx} = \frac{y(2x-y)}{x(2y-x)} = \frac{-1(2+1)}{1(-2-1)} = \frac{-3}{-3} = 1 > 0 \quad \left[\frac{dy}{dx}=1\right]$

5.

$$\frac{dy}{dx} = \frac{2xy-y^2}{2xy-x^2} \Rightarrow \frac{dy}{dx}(1,-1) = \frac{2(1)(-1)-(-1)^2}{2(1)(-1)-(1)^2} = \frac{-2-1}{-2-1} = \frac{-3}{-3} = 1$$

$$\square = 2xy-y^2 \Rightarrow \square(1,-1) = 2(1)(-1)-(-1)^2 = -2-1 = -3$$

$$\Delta = 2xy-x^2 \Rightarrow \Delta(1,-1) = 2(1)(-1)-(1)^2 = -2-1 = -3$$

$$\square' = 2y+2x\left(\frac{dy}{dx}\right)-2y\left(\frac{dy}{dx}\right) \Rightarrow \square'(1,-1) = 2(-1)+2(1)(1)-2(-1)(1) = -2+2+2=2$$

$$\Delta' = 2y+2x\left(\frac{dy}{dx}\right)-2x \Rightarrow \Delta'(1,-1) = 2(-1)+2(1)(1)-2(1) = -2+2-2 = -2$$

$$\therefore \frac{d^2y}{dx^2} = \frac{\square'\Delta-\square\Delta'}{\Delta^2} = \frac{(2)(-3)-(-3)(-2)}{(-3)^2} = \frac{-12}{9} = \frac{-4}{3} \quad \therefore \text{ Concave Down } \left[\frac{d^2y}{dx^2}<0\right]$$

GRAPHING

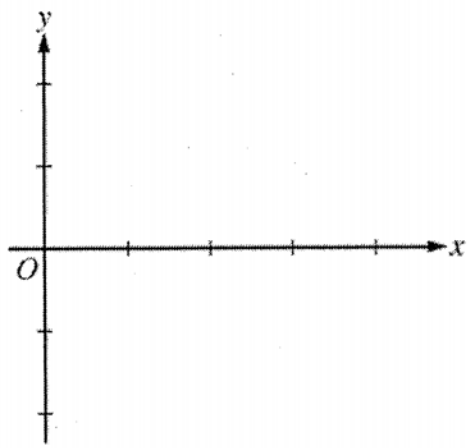
PROBLEM #1 (NO CALCULATOR)

x	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$	3	$3 < x < 4$
$f(x)$	–1	Negative	0	Positive	2	Positive	0	Negative
$f'(x)$	4	Positive	0	Positive	DNE	Negative	–3	Negative
$f''(x)$	–2	Negative	0	Positive	DNE	Negative	0	Positive

Let f be a function that is continuous on the interval $[0, 4)$. The function f is twice differentiable except at $x = 2$. The function f and its derivatives have the properties indicated in the table above, where DNE indicates that the derivatives of f do not exist at $x = 2$.

(a) For $0 < x < 4$, find all values of x at which f has a relative extremum. Determine whether f has a relative maximum or a relative minimum at each of these values. Justify your answer.

(b) On the axes provided, sketch the graph of a function that has all the characteristics of f.



(c) Let g be the function defined by $g(x) = \int_1^x f(t)\,dt$ on the open interval $(0, 4)$. For $0 < x < 4$, find all values of x at which g has a relative extremum. Determine whether g has a relative maximum or a relative minimum at each of these values. Justify your answer.

(d) For the function g defined in part (c), find all values of x, for $0 < x < 4$, at which the graph of g has a point of inflection. Justify your answer.

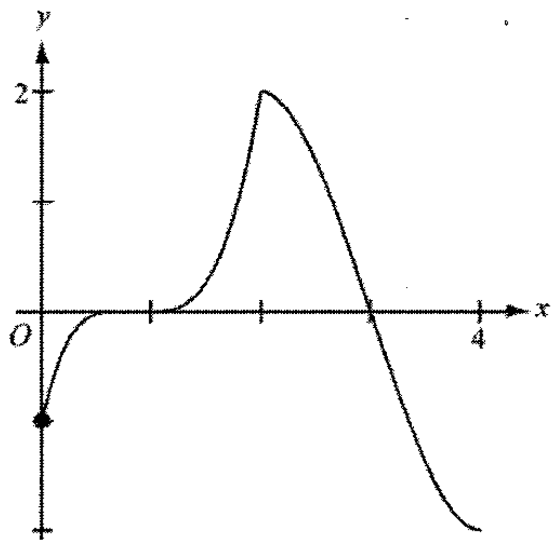
(e) Set up, but do not evaluate, an expression that would result in the area of $f(x)$ for $0 < x < 4$.

GRAPHING ANSWERS

PROBLEM #1 ANSWERS:

(a) f has a relative maximum at $x = 2$ because f' changes from positive to negative at $x = 2$.

(b)



(c) $g'(x) = f(x) = 0$ at $x = 1, 3$.
g' changes from negative to positive at $x = 1$ so g has a relative minimum at $x = 1$. g' changes from positive to negative at $x = 3$ so g has a relative maximum at $x = 3$.

(d) The graph of g has a point of inflection at $x = 2$ because $g'' = f'$ changes sign at $x = 2$.

(e) Area $= -\int_0^1 f(x)\,dx + \int_1^3 f(x)\,dx - \int_3^4 f(x)\,dx$